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Finite-dimensional representations of the quadratic algebra: Applications to the exclusion process

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Abstract. We study the one-dimensional partially asymmetric simple exclusion process (ASEP) with open boundaries, that describes a system of hard-core particles hopping stochastically on a chain coupled to reservoirs at both ends. Derrida and coworkers showed in 1993 that the stationary probability distribution of this model can be represented as a trace on a quadratic algebra, closely related to the deformed oscillator-algebra. We construct *all finite-dimensional* irreducible representations of this algebra. This enables us to compute the stationary bulk density as well as all correlation lengths for the ASEP on a set of special curves of the phase diagram.

1. Introduction

The one-dimensional asymmetric simple exclusion process (ASEP) has been extensively studied by mathematicians [1] and physicists (see [2] and references therein) as one of the simplest model of a system out of equilibrium. It is a diffusion model with hard-core exclusion that can be used to describe hopping conductivity in superionic conductors, traffic flow, and interface growth; it can be mapped to a lattice version of the Kardar–Parisi–Zhang equation [3].

Subject to open boundaries, where particles are injected or extracted, the ASEP displays a rather rich behaviour; it exhibits phase transitions in the thermodynamic limit. The exact steady state for the fully asymmetric case (particles jump only in one direction) was computed in [4] for special choices of the input parameters and more generally in [5, 6]. These computations rely on a recursion in the system size. An elegant way to exploit the recursive property of the steady state is the matrix ansatz used in [5], where the stationary probabilities are expressed as matrix elements of products of operators, which represent particles and holes.

The same technique was used in [7] for the partially asymmetric exclusion process with open boundaries. The operators used here generate a quadratic algebra which is simply related to the so-called deformed oscillator algebra. The phase diagram was derived in the thermodynamic limit, and the steady-state current in each phase was computed. However, since suitable representations of the quadratic algebra were lacking general equal-time correlation functions, the stationary state could not be calculated. Recently, Essler and Rittenberg [8] studied Fock representations of the general quadratic algebra (that

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can generically be mapped on the deformed oscillator algebra). They first found infinite-dimensional matrices, and then gave conditions under which these infinite matrices have an invariant subspace of finite dimension. These constraints were explicitely written in terms of the parameters of the partially ASEP in the case of a one- and two-dimensional invariant subspace. One- and two-dimensional representations were constructed, and correlation functions were determined with the help of these representations.

In this paper we use standard methods of linear algebra to classify and construct all irreducible finite-dimensional representations of the deformed harmonic oscillator algebra, without having to start with infinite-dimensional matrices. We show that there is exactly one irreducible representation for any finite dimension n, and we find explicit constraints on the parameters of the ASEP that make this representation compatible with the boundary conditions (cf equation (20)). These constraints define the locus of points in the phase diagram of the exclusion model that are accessible by finite-dimensional representations. In these regions of the phase diagram we compute the bulk density in thermodynamic limit and all the correlation lengths. Our results prove some conjectures raised in [8]. Morever, to the best of our knowledge the finite-dimensional representation presented here has not been reported before†. The matrices obtained can also be useful in studying algebras associated with more general reaction–diffusion processes.

This paper is organized as follows. In section 2, we recall the matrix technique, relate it to the deformed harmonic oscillator algebra, and construct all irreducible finite-dimensional representations of this algebra. Section 3 is devoted to the calculation of the bulk density and the correlation lengths in the thermodynamic limit in the regions of the phase space that are accessible by finite-dimensional representations. The final section concludes with some remarks on possible generalizations. Some general properties of the quadratic algebra which are shared by all representations are proved in appendix A; these properties are used in appendix B to prove the non-triviality of the representations obtained in section 2.

2. Finite-dimensional representations of the quadratic algebra

2.1. Matrix ansatz for the ASEP with open boundaries

We consider the one-dimensional partially asymmetric exclusion process with open boundary conditions. Each site $i (1 \leqslant i \leqslant L)$ of a one-dimensional lattice of L sites is either occupied by a particle ($\tau_i = 1$) or empty ($\tau_i = 0$). The system evolves according to a stochastic dynamical rule: during each infinitesimal time step dt, the transitions allowed for the bond $(i, i+1)$ with $1 \leqslant i \leqslant L-1$ are

$$\begin{aligned} &10 \to 01 \qquad \text{with rate } 1 \\ &01 \to 10 \qquad \text{with rate } x. \end{aligned} \tag{1}$$

The parameter x is positive and measures the strength of the driving field; one can assume with no restriction that $x < 1$. The model studied here is not totally asymmetric, therefore $x > 0$. Particles are injected at sites 1 and L with rates $(1-x)\alpha$ and $(1-x)\delta$ and extracted with rates $(1-x)\gamma$ and $(1-x)\beta$ respectively, where α, β, γ and δ are strictly positive numbers.

† A reason for this fact may be that, as shown in [5], there are no irreducible representations of finite dimension, strictly greater than 1, of the algebra associated with the totally asymmetric exclusion process, the model mostly studied.

It was shown in [5] that the quantities $f(\tau_1, \ldots, \tau_L)$ defined as

$$f(\tau_1, \ldots, \tau_L) = \left\langle W \middle| \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) \middle| V \right\rangle \tag{2}$$

provide stationary solutions of the master equation of the ASEP if the operators D, E satisfy the algebra

$$DE - xED = (1 - x)(D + E) \tag{3}$$

and the vectors $\langle W|$ and $|V\rangle$ are such that

$$\begin{aligned} (\beta D - \delta E)|V\rangle &= |V\rangle \\ \langle W|(\alpha E - \gamma D) &= \langle W|. \end{aligned} \tag{4}$$

The ASEP on a finite lattice has a unique stationary state (Perron–Frobenius theorem, see, for example [9]). Therefore, all the quantities $f(\tau_1, \ldots, \tau_L)$ are proportional to the steady-state probabilities of the system. If the matrix elements (2) are not all equal to zero, then the normalized stationary probabilities of the exclusion process are given by:

$$P(\tau_1, \ldots, \tau_L) = \frac{1}{Z_L} f(\tau_1, \ldots, \tau_L) \qquad \text{with } Z_L = \langle W|(D + E)^L|V\rangle. \tag{5}$$

To compute the stationary probability distribution of a system with L sites, one needs to find representations of the algebra (3) and boundary vectors (4). These representations must be such that the matrix elements of length L, i.e. of the type

$$\langle W|D^{n_1} E^{m_1} \ldots D^{n_k} E^{m_k}|V\rangle \qquad \text{with } n_1 + m_1 + \cdots + n_k + m_k = L \tag{6}$$

do not identically vanish (i.e. are non-trivial). In appendix A, we find general properties, that a representation must possess to ensure that matrix elements of the type (6) are non-trivial for a system of length L.

In [8], relations (3) and (4) are interpreted as a Fock representation of the quadratic algebra generated by the operators

$$A = \beta D - \delta E - 1 \qquad \text{and} \qquad B = \alpha E - \gamma D - 1. \tag{7}$$

Matrices A and B act trivially on the boundary vectors:

$$A|V\rangle = 0 \qquad \text{and} \qquad \langle W|B = 0. \tag{8}$$

Thus $|V\rangle$ and $\langle W|$ play the role of right and left vacuum for the operators A and B. If $\alpha\beta \neq \gamma\delta$ the operators D and E can be expressed as linear combinations of A and B, which hence satisfy the quadratic algebra:

$$\begin{aligned} &(1 - x)\alpha\gamma A^2 + (\alpha\beta - x\gamma\delta)AB + (\gamma\delta - x\alpha\beta)BA + (1 - x)\delta\beta B^2 \\ &\quad = (1 - x)((\alpha\beta - \gamma\delta)(\alpha + \gamma) - \alpha(\beta + \gamma) - \gamma(\alpha + \delta))A \\ &\qquad + (1 - x)((\alpha\beta - \gamma\delta)(\beta + \delta) - \delta(\beta + \gamma) - \beta(\alpha + \delta))B \\ &\qquad + (1 - x)((\alpha\beta - \gamma\delta)(\alpha + \beta + \gamma + \delta) - (\alpha + \delta)(\beta + \gamma)). \end{aligned} \tag{9}$$

Conversely, a Fock representation of a general quadratic algebra can generically be transformed to a representation of an algebra of type (3) with boundary conditions like (4).

2.2. Classification of irreducible representations

We show that the n-dimensional irreducible representation of the algebra (3) can be written in a suitable basis as:

$$D = \begin{pmatrix} 1+a & 0 & 0 & 0 & . & . \\ 0 & 1+ax & 0 & 0 & & \\ 0 & 0 & 1+ax^2 & 0 & & \\ & & & . & & \\ & & & & . & \\ & & & & & 1+ax^{n-1} \end{pmatrix} \tag{10}$$

and

$$E = \begin{pmatrix} 1+\frac{1}{a} & 0 & 0 & 0 & . & . \\ 1 & 1+\frac{1}{ax} & 0 & 0 & & \\ 0 & 1 & 1+\frac{1}{ax^2} & 0 & & \\ & & . & . & & \\ & & & . & . & \\ & & & & 1 & 1+\frac{1}{ax^{n-1}} \end{pmatrix} \tag{11}$$

where a is a non-zero real parameter.

In order to derive this result, we start by defining two operators d and e such that

$$D = 1 + d \qquad \text{and} \qquad E = 1 + e. \tag{12}$$

From equation (3), we see that d and e satisfy the deformed harmonic oscillator algebra [10, 11]:

$$de - xed = 1 - x. \tag{13}$$

Using equation (13), we note that if λ is an eigenvalue of the operator de, then

$$\left(1 + \frac{\lambda - 1}{x}\right)$$

is an eigenvalue of the operator ed (we recall that $x \neq 0$). In finite dimension the matrices de and ed have the same spectrum, which contains therefore the numbers:

$$\lambda, 1 + \frac{\lambda - 1}{x}, 1 + \frac{\lambda - 1}{x^2}, \ldots, 1 + \frac{\lambda - 1}{x^k}, \ldots k \in \mathbb{N}.$$

But there is only a finite number of distinct eigenvalues in a finite-dimensional representation: this implies that either x is a root of unity or $\lambda = 1$. Only the latter case is possible here: the eigenvalues of de must all be equal to 1. The operator de is invertible, and so are the matrices d and e (their determinant cannot be zero). We now rewrite equation (13) as:

$$d(e - d^{-1}) = x(e - d^{-1})d. \tag{14}$$

Hence, our problem is reduced to finding representations of the algebra

$$\mathcal{D}\mathcal{E} = x\mathcal{E}\mathcal{D} \qquad \text{with } \mathcal{D} \text{ invertible}$$

where

$$\mathcal{D} = d \qquad \text{and} \qquad \mathcal{E} = (e - d^{-1}).$$

If $|a\rangle$ is an eigenvector of $\mathcal{D}$ with eigenvalue a (which is different from 0 because D is invertible), then $\mathcal{E}|a\rangle$ is an eigenvector of $\mathcal{D}$ with eigenvalue xa or is the null vector. Consequently, the space spanned by the linearly independent vectors $\{|a\rangle, \mathcal{E}|a\rangle, \ldots, \mathcal{E}^k|a\rangle, \ldots\}$ is stable under both $\mathcal{D}$ and $\mathcal{E}$. To obtain a finite-dimensional representation, there must be an integer n such that $\mathcal{E}^n|a\rangle = 0$. This integer n is also the dimension of the full representation space, since the representation is irreducible.

The proof that the irreducible n-dimensional representation of the quadra-tic oscillator algebra is given by (10) and (11), is completed by writing the matrices of $D = 1+d = 1+\mathcal{D}$ and $E = 1 + e = 1 + \mathcal{E} + \mathcal{D}^{-1}$ in the basis $(|a\rangle, \mathcal{E}|a\rangle, \ldots, \mathcal{E}^{n-1}|a\rangle)$. (Hereafter, this basis will be denoted by $(|1\rangle, \ldots, |n\rangle)$.)

2.3. Boundary vectors and conditions on the rates

We construct the boundary vectors $|V\rangle$ and $\langle W|$ associated with the irreducible representation of dimension n found above. These boundary vectors are right and left eigenvectors of the operators A and B, defined by equations (7), with zero eigenvalue (see equation (8)). In the basis $(|1\rangle, \ldots, |n\rangle)$ matrices A and B are bidiagonal, their eigenvalues are readily obtained. One can check that for A and B to have 0 as an eigenvalue, there must be two integers, k and l, between 0 and $n-1$ such that:

$$\beta(1 + ax^k) - \delta\left(1 + \frac{1}{ax^k}\right) - 1 = 0$$

and

$$\alpha\left(1 + \frac{1}{ax^l}\right) - \gamma(1 + ax^l) - 1 = 0. \tag{15}$$

Necessarily $|l - k| = n - 1$, otherwise a representation of dimensions less than n would suffice. In the case $k = n - 1$ and $l = 0$, one has

$$|V\rangle = |n\rangle \qquad \text{and} \qquad \langle W| = \langle 1|.$$

With such a choice, one has $\langle W|D^L|V\rangle = 0$ for any L. This implies, using the definiteness property (proved in appendix A), that all the quantities $f(\tau_1, \ldots, \tau_L)$ are equal to 0: this case has to be excluded.

The only case that remains is $k = 0$ and $l = n - 1$; the two equations corresponding to (15) are

$$\begin{aligned} &\beta a^2 + (\beta - \delta - 1)a - \delta = 0 \\ &\alpha\left(\frac{1}{ax^{n-1}}\right)^2 + (\alpha - \gamma - 1)\left(\frac{1}{ax^{n-1}}\right) - \gamma = 0 \end{aligned} \tag{16}$$

and they must have a common root a. Both equations have the same structure, their solutions can be written as $a = \kappa_\pm(\beta, \delta)$ and $\frac{1}{ax^{n-1}} = \kappa_\pm(\alpha, \gamma)$ with

$$\kappa_\pm(u, v) = \frac{-u + v + 1 \pm \sqrt{(u - v - 1)^2 + 4uv}}{2u}. \tag{17}$$

This function already appeared in the study of the phase diagram of the partially asymmetric exclusion process in [7]. One can check that for u and v positive,

$$\kappa_+(u, v) > 0 \qquad \text{and} \qquad -1 < \kappa_-(u, v) < 0. \tag{18}$$

One also has

$$\kappa_+(u, v)\kappa_-(u, v) = -\frac{v}{u}. \tag{19}$$

Therefore, equations (16) have a common root if and only if

$$x^{1-n} = \kappa_+(\beta, \delta)\kappa_+(\alpha, \gamma). \tag{20}$$

Condition (20) is an explicit constraint on the parameters of the ASEP. It defines the locus of points in the phase diagram accessible to n-dimensional representations. Figure 1 shows the curves (hyperbolae branches) in the $\kappa_+(\alpha, \gamma) - \kappa_+(\beta, \delta)$ plane where $n = 1, \ldots, 8$-dimensional representations exist. It is worthwhile noticing that equation (20) proves a conjecture made in [8], which states that finite-dimensional representations exist only in the region

$$\kappa_+(\beta, \delta)\kappa_+(\alpha, \gamma) > 1.$$

If condition (20) is fulfilled the common root of (16) reads

$$a = \kappa_+(\beta, \delta) = \frac{1}{x^{n-1}\kappa_+(\alpha, \gamma)} \tag{21}$$

and vectors $\langle W|$ and $|V\rangle$ exist such that boundary conditions (4) are satisfied. These vectors can be computed as:

$$\langle W| = (w_1, w_2, \ldots, w_n) \qquad |V\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix} \tag{22}$$

with

$$v_k = \prod_{m=1}^{k-1} \frac{\delta}{\beta[1 - x^{-m}][ax^m - \kappa_-(\beta, \delta)]} \tag{23}$$

and

$$w_k = \prod_{m=k-1}^{n-2} \frac{-1}{[1 - x^{m-n+1}][(ax^m)^{-1} - \kappa_-(\alpha, \gamma)]} \qquad \text{for } k = 1, \ldots, n. \tag{24}$$

With this definition, one has $v_1 = 1$ and $w_n = 1$.

Hence, we have constructed an explicit n-dimensional representation ((10), (11), (22), (23) and (24)) of the quadratic algebra with the required boundary conditions, provided that the rates α, β, γ and δ satisfy constraint (20). As discussed earlier (section 2.1), we must verify that the matrix elements of length L (6), computed with this representation, do not identically vanish. In appendix B, using general properties of the algebra derived in appendix A, we prove that these matrix elements are non-trivial if the size L of the system is larger than the dimension n of the matrices. For $L < n$ there are certain choices of the parameters for which, surprisingly enough, all matrix elements of length L vanish. We identify these choices of the parameters in appendix B. In any case, our representation can be used to investigate thermodynamic behaviour.

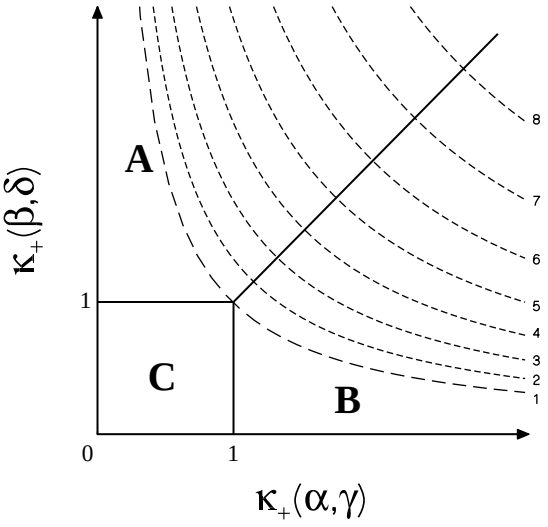

Figure 1. Phase diagram for the ASEP with $0 < x < 1$ in terms of $\kappa_+(\alpha, \gamma)$ and $\kappa_+(\beta, \delta)$ where $\kappa_+(u, v) = \frac{1}{2u}[-u + v + 1 \pm \sqrt{(u - v - 1)^2 + 4uv}]$. The phases are separated by the full lines. The broken curves are where finite-dimensional representations exist for $x = \frac{3}{4}$, and the numbers attached to these curves are the dimensions of the corresponding representations.

3. Bulk density and correlations in the thermodynamic limit

In the thermodynamic limit the asymmetric exclusion process exhibits three different phases in which the current and correlation functions are given by different expressions. An exhaustive study has been carried out, mainly for the totally asymmetric model [5, 6, 12]. Much less is known about the partially asymmetric exclusion process: the phase diagram was obtained in [7] from an exact calculation of the current in the thermodynamic limit. This phase diagram, shown in figure 1, agrees with the mean-field prediction. The bulk density was computed in [8] in mean-field approximation, and it was argued that the mean-field result was exact. We recall here the description of the phase diagram together with some known results.

The phases can be described as follows.

• Phase A (high density): $\kappa_+(\beta, \delta) > \kappa_+(\alpha, \gamma)$ and $\kappa_+(\beta, \delta) > 1$.

In the thermodynamic limit, the current J_A is

$$J_{\mathrm{A}} = (1 - x)\frac{\kappa_+(\beta, \delta)}{[1 + \kappa_+(\beta, \delta)]^2}$$

and the mean-field prediction for the density in the bulk is:

$$\rho_{\mathrm{A}}^{\mathrm{MF}} = \frac{\kappa_+(\beta, \delta)}{1 + \kappa_+(\beta, \delta)}.$$

• Phase B (low density): $\kappa_+(\alpha, \gamma) > \kappa_+(\beta, \delta)$ and $\kappa_+(\alpha, \gamma) > 1$.

$$J_{\mathrm{B}} = (1 - x)\frac{\kappa_+(\alpha, \gamma)}{[1 + \kappa_+(\alpha, \gamma)]^2}$$

$$\rho_{\mathrm{B}}^{\mathrm{MF}} = \frac{1}{1 + \kappa_+(\alpha, \gamma)}.$$

- Phase C (maximal current): $\kappa_+(\beta,\delta) < 1$ and $\kappa_+(\alpha,\gamma) < 1$.

$$J_{\mathrm{C}} = \frac{1-x}{4}$$
$$\rho_{\mathrm{C}}^{\mathrm{MF}} = \tfrac{1}{2}.$$

Phases A and B are separated by a coexistence line, defined by $\kappa_+(\beta,\delta) = \kappa_+(\alpha,\gamma) > 1$, through which the (mean-field) bulk density is discontinuous. This phenomenon has been investigated in the totally asymmetric case, and reveals the presence of a shock between a region of low density and a region of high density [5, 6]. The same phenomenon happens in the partially asymmetric case, as will be confirmed by our calculation of correlation functions.

Little is known about general correlation functions of the partially asymmetric exclusion process. In [8], some correlation functions were computed with one- and two-dimensional representations, i.e. on curves 1 and 2 in figure 1. We shall now identify the generic structure of correlation functions for the case where n-dimensional representations exist. Such an exact result cannot be obtained from a mean-field analysis [5, 8]. For example a two-point function is given by:

$$\langle \tau_j \tau_k \rangle = \frac{\langle W|C^{j-1}DC^{k-j-1}DC^{L-k}|V\rangle}{\langle W|C^{L}|V\rangle} \tag{25}$$

with

$$C = D + E. \tag{26}$$

The positions, on which a correlation function depends, enter the above expression as an exponent of the matrix C. For this reason we have to compute powers of the matrix C which is given by:

$$C = D + E = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & . & . \\ 1 & \lambda_2 & 0 & 0 & & \\ 0 & 1 & \lambda_3 & 0 & & \\ & & . & . & & \\ & & & . & . & \\ & & & & 1 & \lambda_n \end{pmatrix} \tag{27}$$

with

$$\begin{aligned} \lambda_k &= 2 + ax^{k-1} + \frac{1}{ax^{k-1}} \\ &= 2 + x^{k-1}\kappa_+(\beta,\delta) + x^{n-k}\kappa_+(\alpha,\gamma). \end{aligned} \tag{28}$$

We have used equations (20) and (21) to derive the last line.

For $L \gg 1$, the dominating matrix elements of C^L will be found in the invariant subspace of C that corresponds to the largest eigenvalue. We must identify the highest eigenvalue, and examine whether C can be diagonalized in the associated invariant space†. We notice that all the eigenvalues of C lie on the curve $z \mapsto 2 + z + 1/z$, with $z = a, ax, \ldots, ax^{n-1}$. This implies that the largest eigenvalue of C can only be λ_1, or λ_n or both of them if it happens that $\lambda_1 = \lambda_n$. We have:

$$\begin{aligned} \lambda_1 - \lambda_n &= a + \frac{1}{a} - \left(ax^{n-1} + \frac{1}{ax^{n-1}} \right) = (1 - x^{n-1}) \left(a - \frac{1}{ax^{n-1}} \right) \\ &= (1 - x^{n-1})(\kappa_+(\beta,\delta) - \kappa_+(\alpha,\gamma)). \end{aligned} \tag{29}$$

† We recall that the invariant space associated with the eigenvalue λ is given by $\bigcup_{i=0}^{\infty} \mathrm{Ker}(C-\lambda)^i$.

There are three different cases to consider, that naturally correspond to different regions in the phase space.

3.1. Bulk density and correlations in phase A

This phase is defined by $\kappa_+(\beta,\delta) > \kappa_+(\alpha,\gamma)$ and $\kappa_+(\beta,\delta) > 1$, therefore $\lambda_1 > \lambda_n$. The leading eigenvalue of C is λ_1 and it is non-degenerate. We write $|\lambda_1\rangle$ (and $\langle\lambda_1|$) for the corresponding right (and left) eigenvector. One should notice that $\langle\lambda_1|$ is equal to $\langle 1|$. Hence, we obtain that for large system sizes:

$$Z_L = \langle W|C^L|V\rangle \simeq \langle W|\lambda_1\rangle\langle\lambda_1|V\rangle\lambda_1^L. \tag{30}$$

The bulk density is given by:

$$\begin{aligned}\rho_{\mathrm{A}} = \langle\tau_i\rangle &= \frac{\langle W|C^{i-1}DC^{L-i}|V\rangle}{\langle W|C^L|V\rangle} \qquad \text{with } 1 \ll i \ll L \\ &\simeq \frac{\langle W|\lambda_1\rangle\lambda_1^{i-1}\langle\lambda_1|D|\lambda_1\rangle\lambda_1^{L-i}\langle\lambda_1|V\rangle}{\langle W|\lambda_1\rangle\langle\lambda_1|V\rangle\lambda_1^L} \\ &= \frac{1+a}{\lambda_1} \qquad (31) \\ &= \frac{\kappa_+(\beta,\delta)}{1+\kappa_+(\beta,\delta)}. \qquad (32)\end{aligned}$$

Here we used $\langle\lambda_1|D = \langle\lambda_1|(1+a)$ and $\langle\lambda_1|\lambda_1\rangle = 1$ to derive the last two lines. This result agrees with the mean-field prediction, as it was conjectured in [8].

Next we compute $\langle W|C^{j-1}DC^{k-j-1}DC^{L-k}|V\rangle$. The result is a linear combination of powers of the eigenvalues. Dividing it by the asymptotic expression (30) for Z_L and using equation (25) shows that the two-point function is a superposition of terms $(\lambda_i/\lambda_1)^m$, where m is one of the distances $j, k{-}j, L{-}k$. Similarly, one can compute higher-order correlation functions. For a large system, i.e. for $L \gg n$, they decay exponentially with correlation lengths

$$\begin{aligned}\xi_i &= \left\{\ln\frac{\lambda_1}{\lambda_i}\right\}^{-1} \\ &= \left\{\ln\frac{2+x^{n-1}\kappa_+(\alpha,\gamma)+\kappa_+(\beta,\delta)}{2+x^{i-1}\kappa_+(\beta,\delta)+x^{n-i}\kappa_+(\alpha,\gamma)}\right\}^{-1} \qquad i = 1,\ldots,n-1. \qquad (33)\end{aligned}$$

For $n = 2$ the correlation length computed in [8] is recovered. One should notice that the correlation lengths depend only on the functions $\kappa_+(\beta,\delta)$ and $\kappa_+(\alpha,\gamma)$.

3.2. Bulk density and correlations in phase B

Phase B, defined by $\kappa_+(\alpha,\gamma) > \kappa_+(\beta,\delta)$ and $\kappa_+(\alpha,\gamma) > 1$, is related to phase A by means of a symmetry: replacing τ_i by $1-\tau_{L+1-i}$ as well as β by α, δ by γ and vice versa leaves the dynamics of the system invariant. We find that the bulk density is:

$$\begin{aligned}\rho_{\mathrm{B}} &= \frac{1+ax^{n-1}}{\lambda_n} \qquad (34) \\ &= \frac{1}{1+\kappa_+(\alpha,\gamma)}. \qquad (35)\end{aligned}$$

And the correlation lengths are given by:

$$\xi_i = \left\{ \ln \frac{2 + x^{n-1}\kappa_+(\beta,\delta) + \kappa_+(\alpha,\gamma)}{2 + x^{i-1}\kappa_+(\beta,\delta) + x^{n-i}\kappa_+(\alpha,\gamma)} \right\}^{-1} \qquad i = 1, \ldots, n-1. \quad (36)$$

3.3. Correlation functions on the coexistence line

Phases A and B coexist in a single system on the line $\kappa_+(\beta,\delta) = \kappa_+(\alpha,\gamma) > 1$. For the fully asymmetric process it was shown in [5, 6] that the density profile depends linearly on the position, which indicates that the border between the two phases can be anywhere with the same probability. The same behaviour was observed in [8] for the case where two-dimensional representations exist, and two-point correlations functions were found to depend algebraically on the positions.

Let us discuss the case where n-dimensional representations exist. The eigenvalues of C are given by equation (28) which now reads

$$\lambda_k = 2 + [x^{k-1} + x^{n-k}]\kappa_+(\beta,\delta) \quad (37)$$

because $\kappa_+(\alpha,\gamma) = \kappa_+(\beta,\delta)$. Obviously, these eigenvalues pairwise coincide: $\lambda_k = \lambda_{n+1-k}$ for all k, and due to the line under the diagonal (see equation (27)) C is not diagonalizable. However, it can be transformed into a Jordan normal form with a two dimensional Jordan block for each eigenvalue. This implies algebraic behaviour of the correlation functions (see the discussion in [13]). We denote by $(|\lambda_1\rangle, |\lambda_n\rangle)$ a basis of the two-dimensional invariant space associated with the highest eigenvalue $\lambda = \lambda_1 = \lambda_n$. In this basis the restriction of C^L can be written as:

$$\lambda^{L-1} \begin{pmatrix} \lambda & 0 \\ L & \lambda \end{pmatrix}. \quad (38)$$

We choose $|\lambda_n\rangle = |n\rangle$ and $\langle\lambda_1| = \langle 1|$. By decomposing on this normal basis, we find for large system sizes

$$Z_L = \langle W|C^L|V\rangle \simeq L\lambda^{L-1} \quad (39)$$

because $w_n = v_1 = 1$.

The m-point correlation function can now be computed as

$$\langle \tau_{x_1 L}\tau_{x_2 L}\ldots\tau_{x_m L}\rangle = \frac{\langle W|C^{x_1 L - x_0 L - 1}DC^{x_2 L - x_1 L - 1}D\ldots DC^{L - x_m L}|V\rangle}{\langle W|C^L|V\rangle} \quad (40)$$

with $0 = x_0 < x_1 < x_2 < \cdots < x_m < 1$. In the limit of large system size, we project on the $(|\lambda_1\rangle, |\lambda_n\rangle)$ plane and keep only highest-order terms:

$$\begin{aligned} &\langle \tau_{x_1 L}\tau_{x_2 L}\ldots\tau_{x_m L}\rangle \\ &\quad \simeq \sum_{k=1}^{m} \frac{\langle V|C^{x_1 L - 1}D\ldots D|\lambda_n\rangle\langle\lambda_n|C^{x_k L - x_{k-1} L - 1}|\lambda_1\rangle\langle\lambda_1|D\ldots DC^{L - x_m L}|V\rangle}{L\lambda^{L-1}} \\ &\quad \simeq \sum_{k=1}^{m} (x_k - x_{k-1}) \frac{(1 + ax^{n-1})^{k-1}(1+a)^{m-k+1}}{\lambda^m}. \end{aligned} \quad (41)$$

To obtain the last line we used that $D|\lambda_n\rangle = (1 + ax^{n-1})|\lambda_n\rangle$ and $\langle\lambda_1|D = \langle\lambda_1|(1 + a)$. Using, furthermore, expressions (31) and (34) for the bulk densities in phases A and B we find:

$$\langle \tau_{x_1 L}\tau_{x_2 L}\ldots\tau_{x_m L}\rangle \simeq \sum_{k=1}^{m} (x_k - x_{k-1})\rho_{\mathrm{B}}^{k-1}\rho_{\mathrm{A}}^{m-k+1}. \quad (42)$$

Formula (42) clearly shows that on the coexistence line there is a shock between a low-density phase ρ_B and a high-density phase ρ_A. This shock can be anywhere between 1 and L with the same probability.

4. Conclusion

The finite-dimensional Fock representations of the quadratic algebra, obtained here, allow us to derive exact results for the partially asymmetric exclusion process on some special curves of the phase diagram. We believe that the formulae obtained for the correlation functions are valid throughout phases A and B, but this is a conjecture: in order to prove it, one would have to use infinite-dimensional representations, that would lead to complicated calculations. Maybe a kind of 'analytic continuation argument' could be used to obtain results valid on any curve of the form $\kappa_+(\beta,\delta)\kappa_+(\alpha,\gamma) = x^{1-n}$ where n is not necessarily an integer any more, but we do not know how to do that. An interesting, and related, question is to understand the intuitive physics behind condition (20): What makes a system, that can be analysed with finite $n \times n$ matrices, different from the others?

The matrices that we have constructed are also useful in the representation theory of the algebras associated with more general reaction–diffusion processes. In [8] the following system of algebraic relations is derived for such algebras:

$$\begin{aligned} \kappa_1 DE + \kappa_2 ED &= D + E \\ \kappa_3 D^2 &= \kappa_4 DE + \kappa_5 ED \\ \kappa_6 E^2 &= \kappa_7 DE + \kappa_8 ED \end{aligned} \tag{43}$$

where the κ_j can be computed from the rates of the reaction–diffusion process [8]. The representations of (43) are a subset of those of the deformed oscillator algebra. An analysis, similar to those given in section 2.3, would give conditions on the rates for a finite-dimensional representation of (43) to exist.

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Appendix A

In this appendix we show, without using any explicit representation, how to compute recursively expressions of the type

$$\langle W|D^{n_1}E^{m_1}\dots D^{n_k}E^{m_k}|V\rangle \qquad \text{with } n_1 + m_1 + \dots + n_k + m_k = L$$

that we shall call *matrix elements of length* L, and investigate conditions under which these quantities are different from zero. We need the following properties.

Definiteness property. Matrix elements of length L are either all strictly positive, or all strictly negative or all identically equal to zero.

Indeed, matrix elements of length L are stationary solution of the master equation of the ASEP on a open chain with L sites [5]. Therefore, according to the Perron–Frobenius theorem [9], they are unique up to a multiplicative constant and are all of the same sign.

Reordering property. The expression

$$(D^{n_1}E^{m_1}\dots D^{n_k}E^{m_k}) - x^q E^{\sum_{i=1}^k m_i} D^{\sum_{j=1}^k n_j}$$

with $L = n_1 + m_1 + \cdots + n_k + m_k$ and $q = \sum_{j\leqslant i\leqslant k} n_j m_i$ is equal to a linear combination of products of $L-1$ operators D or E with strictly positive coefficients.

One begins with the formula (obtained by induction on m_1):

$$DE^{m_1} = x^{m_1}E^{m_1}D + (1-x)\left(\frac{1-x^{m_1}}{1-x}E^{m_1} + \sum_{k=0}^{m_1-1} x^k E^k D E^{m_1-1-k}\right)$$

that allows us to transfer a matrix D from the left to the right of E^{m_1}, and proves the reordering property for $DE^{m_1} - x^{m_1}E^{m_1}D$. One concludes then by induction on k and n_k.

The reordering property enables us to compute all matrix elements of length L if expressions of the type $\langle W|E^l D^{L-l}|V\rangle$ with $l = 0,\dots,L$ and all matrix elements of length $L-1$ are known. Hence, one can relate matrix elements of length L to matrix elements of length $L-1$. This can be achieved starting with the following system which is a simple consequence of (4):

$$\begin{cases} \alpha\langle W|E^{l+1}D^{L-l-1}|V\rangle - \gamma\langle W|DE^l D^{L-l-1}|V\rangle = \langle W|E^l D^{L-l-1}|V\rangle \\ -\delta\langle W|E^l D^{L-l-1}E|V\rangle + \beta\langle W|E^l D^{L-l}|V\rangle = \langle W|E^l D^{L-l-1}|V\rangle. \end{cases} \tag{A1}$$

This system is rewritten according to the reordering property:

$$\begin{cases} \alpha\langle W|E^{l+1}D^{L-l-1}|V\rangle - x^l\gamma\langle W|E^l D^{L-l}|V\rangle = \mathcal{W}_{L-1} \\ -x^{L-l-1}\delta\langle W|E^{l+1}D^{L-l-1}|V\rangle + \beta\langle W|E^l D^{L-l}|V\rangle = \mathcal{W}'_{L-1} \end{cases} \tag{A2}$$

where $\mathcal{W}_{L-1}$ and $\mathcal{W}'_{L-1}$ are positive linear combinations of matrix element of length $L-1$. We must distinguish two cases:

(i) if $\alpha\beta - x^{L-1}\gamma\delta \neq 0$, then *all expressions of length L can be computed from expressions of length* $L-1$. Furthermore, all matrix elements of length L are equal to 0 if and only if all those of length $L-1$ are equal to zero.

(ii) If $\alpha\beta - x^{L-1}\gamma\delta = 0$, then *all matrix elements of lengths less than or equal to* $L-1$ *identically vanish*. Indeed, the l.h.s. of the equations in (A2) are proportional, therefore one must have:

$$0 = \begin{vmatrix} \alpha & \mathcal{W}_{L-1} \\ -x^{L-l-1}\delta & \mathcal{W}'_{L-1} \end{vmatrix} = \alpha\mathcal{W}'_{L-1} + x^{L-l-1}\delta\mathcal{W}_{L-1}. \tag{A3}$$

Because of the definiteness property the positive linear combination on the r.h.s. can only be equal to 0, if all matrix elements of length $L-1$ are equal to 0. For any $n \leqslant L-2$, one has $\alpha\beta - x^n\gamma\delta \neq 0$, because otherwise this would imply $\alpha\beta = \gamma\delta = 0$ which is impossible since all the rates are strictly positive. Using case (i), one concludes that all matrix elements of lengths less than or equal to $L-1$ are identically 0.

In conclusion, we have proved the following properties of any representation of algebra (3) together with boundary conditions (4).

Property A. If there is one matrix element of length L different from 0, then all matrix elements of lengths greater than or equal to L are non-zero.

Property B. Suppose that there is one matrix element of lengths less than L different from 0, then for expressions of lengths less than or equal to L, one has to consider two cases:

(i) if for all non-negative integer l, $\alpha\beta - x^l\gamma\delta \neq 0$, then all the expressions $\langle W|D^{n_1}E^{m_1}\ldots D^{n_k}E^{m_k}|V\rangle$ are non-zero for any system size.

(ii) If there exists one l such that $\alpha\beta - x^l\gamma\delta = 0$, then all expressions of lengths less than or equal to l identically vanish but all matrix elements of lengths greater than l are non-zero.

Appendix B

We first show, using property A derived in appendix A, that the representations found in sections 2.2 and 2.3 provide non-trivial weights for systems of lengths greater than or equal to n. Indeed the matrix elements $\langle W|D^l|V\rangle$ for $l = 0, \ldots, n-1$ cannot all vanish, otherwise one would have:

$$0 = \langle W|D^l|V\rangle = \sum_{i=1}^{n} (\mu_i)^l w_i v_i \qquad \text{for } l = 0, \ldots, n-1 \tag{B1}$$

where $\mu_i = 1 + ax^{i-1}$ is the ith eigenvalue of D. But relations (B1) can be interpreted as a system of n equations with n unknowns $(w_i v_i)$. This system is a Van der Monde system, and as all the eigenvalues μ_i are different from each other, the only solution is $(w_i v_i) = 0$ for $i = 1, \ldots, n$. But this is not the case: as one can see from the explicit formulae ((23) and (24)), all the components of $|V\rangle$ and $\langle W|$ are different from 0. Consequently, the matrix elements $\langle W|D^l|V\rangle$ cannot vanish for all l, i.e. there is an $l' < n$ such that $\langle W|D^{l'}|V\rangle \neq 0$. According to property A, of appendix A, we conclude that representations (10) and (11), together with the boundary vectors given by (22) and (23) always provide the stationary probabilities of the partially asymmetric exclusion process for systems of sizes bigger than n.

Secondly, for systems of sizes strictly less than n, one has to check, according to property B of appendix A, whether $\alpha\beta - x^k\gamma\delta$ can be equal to 0 for some non-negative integer k. Using (19) and (20), this condition is equivalent to:

$$x^{n-1-k} = \kappa_-(\beta, \delta)\kappa_-(\alpha, \gamma). \tag{B2}$$

So, if (B2) is never satisfied, the n-dimensional representation found above allows us to compute the stationary weights for any system size. If there is a k_0, such that (B2) is true, then k_0 has to be less than $n-2$ (because of (18)), and the n-dimensional representation provides non-trivial weights only for systems of sizes bigger than or equal to $k_0 + 1$.

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